ON THE DIFFERENTIABILITY STRUCTURE OF REAL FUNCTIONS

BY

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1. **Introduction.** Let f be a real valued function defined on a set S of real numbers. A subset T of S is called a *differentiable road* for f if the restriction of f to the set T is differentiable. For our purposes, it is convenient to admit points where the derivative is infinite as points of differentiability.

The major purpose of this article is to study the structure of differentiable roads for functions defined on perfect sets. Our main result is obtained in §3. It asserts that every function defined and continuous on a perfect set P, possesses a perfect differentiable road. Then, in §4, we show that this result can in some ways be extended, while in other ways is the best possible.

In the process of obtaining the results mentioned above, we obtain several other results. In particular, in §2, we obtain a lemma which is of fundamental importance in the proof of the main theorem, but which appears to be of independent interest. This lemma states, roughly, that if every interval contiguous to a perfect nowhere dense set P is colored either black or white, and the endpoints of those intervals colored black form a set somewhere dense in P, then there exists a perfect set Q contained in P such that if x and y are arbitrary points of Q, the longest of the colored intervals contained in the interval determined by x and y is black. We use this lemma in the proof of the main theorem, but it can be used in other ways as well. For example, we use it to obtain a relatively simple proof of a theorem of Filipczak [4], according to which a function continuous on a perfect set P is necessarily monotonic on some perfect subset of P.

2. **Preliminary results.** In this section we develop some of the machinery which we shall need in the sequel. We begin with the lemma mentioned in the Introduction, which we will refer to as the Black and White Lemma. We denote by P^* the two-sided limit points of the set P.

Lemma 2.1. *Let P be a nowhere dense perfect set of real numbers. Let $\mathscr{A}$ be the family of intervals complementary to P. Suppose $\mathscr{B}$ is any subfamily of $\mathscr{A}$, the endpoints of whose intervals are somewhere dense in P. Then, there exists a perfect subset Q of P such that whenever x and y are distinct points of Q, the longest interval of $\mathscr{A}$ between x and y belongs to $\mathscr{B}$.*

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Proof. We may assume without loss of generality that P is bounded and the set of endpoints of intervals in $\mathscr{B}$ is dense in all of P. Let (a_1, b_1) be a longest interval in $\mathscr{B}$, and choose $c_1, d_1 \in P^*$, with $c_1 < a_1 < b_1 < d_1$ and $(d_1 - c_1) < 2(b_1 - a_1)$. Let $A_1 = [c_1, a_1] \cap P$, $A_2 = [b_1, d_1] \cap P$. Observe that if $x \in A_1$, $y \in A_2$, then the longest interval of $\mathscr{A}$ in (x, y) is necessarily in $\mathscr{B}$.

Since the endpoints of $\mathscr{B}$ form a set which is dense in P, the interval (c_1, a_1) contains some interval of $\mathscr{B}$. Let (a_2, b_2) be such an interval. Pick $c_2, d_2 \in P^*$ such that $c_1 < c_2 < a_2 < b_2 < d_2 < a_1$ and $(d_2 - c_2) < 2(b_2 - a_2)$. Let $A_{1,1} = [c_2, a_2] \cap P$, $A_{1,2} = [b_2, d_2] \cap P$. Similarly, construct sets $A_{2,1}, A_{2,2}$ from A_2 by choosing a largest interval of $\mathscr{B}$ inside (b_1, d_1) and proceeding as before. Notice that $A_{1,i} \subset A_1$, $A_{2,i} \subset A_2$, $i = 1, 2$, and that if $x \in A_{j,1}$, $y \in A_{j,2}$, then the largest interval of $\mathscr{A}$ in (x, y) belongs to $\mathscr{B}$.

We continue this process inductively for k and obtain a system of closed sets $\{A_{n_1, n_2, \ldots, n_k}\}$ where n ranges over the collection, N, of all functions from the natural numbers to $\{1, 2\}$. This system satisfies

(1) $A_{n_1, n_2, \ldots, n_k, i} \subset A_{n_1, \ldots, n_k}$, $i = 1, 2$.

(2) The diameter of $A_{n_1, \ldots, n_k}$ tends to zero with k.

(3) If $x \in A_{n_1, \ldots, n_k, 1}$, $y \in A_{n_1, \ldots, n_k, 2}$, then the longest interval of $\mathscr{A}$ between x and y is in $\mathscr{B}$.

For each k let $B_k = \bigcup \{A_{n_1, \ldots, n_k} : n \in N\}$. Then, each B_k is closed and by (1) the sequence, $\{B_k\}$, is nested. Define $Q = \bigcap_{k=1}^{\infty} B_k$. Since $P \cap [c_1, d_1]$ is compact, Q is nonempty and closed. We claim that Q is dense-in-itself, hence, perfect; and, that Q satisfies the conclusion of the lemma.

Let $x \in Q$. Then, for some $n \in N$, $x \in \bigcap_{k=1}^{\infty} A_{n_1, \ldots, n_k}$. Let $\varepsilon > 0$. By (2) above we may choose $A_{n_1, \ldots, n_k}$ of diameter less than ε. Let $m \neq n_{k+1}$ and let

$$y \in \bigcap_{i=k+2}^{\infty} A_{n_1, \ldots, n_k, m, n_{k+2}, \ldots, n_i}.$$

Then $y \in Q \cap A_{n_1, \ldots, n_k} \cap A_{n_1, \ldots, n_k, m}$ so that $|x - y| < \varepsilon$ while $x \neq y$. Hence, Q is dense-in-itself.

Finally, let $x, y \in Q$. By (2), for each $n \in N$, $\bigcap_{k=1}^{\infty} A_{n_1, \ldots, n_k}$ is a singleton set. Thus, for some $n, m \in N$ with $n \neq m$ we have $x \in \bigcap_{i=1}^{\infty} A_{n_1, \ldots, n_i}$ and

$$y \in \bigcap_{i=1}^{\infty} A_{m_1, \ldots, m_i}.$$

Let k be the first index for which $n_k \neq m_k$, say, $n_k = 1$, $m_k = 2$. Then, $x \in A_{n_1, \ldots, n_{k-1}, 1}$, $y \in A_{n_1, \ldots, n_{k-1}, 2}$ so that by (3) above the longest interval of $\mathscr{A}$ in (x, y) is in the prescribed family $\mathscr{B}$.

Although we shall not need them in what follows, we make some remarks on this lemma. For example, the measure assigned to each interval of $\mathscr{A}$ need not be its length. It is sufficient to assign to each interval I in $\mathscr{A}$ a positive number in an "upper semicontinuous fashion"; namely, so that the numbers assigned to each interval of $\mathscr{A}$ in some neighborhood of I do not exceed that assigned to I.

A second more explicit example makes use of the fact that the closures of the complementary intervals, $\mathscr{A}$, of P are order isomorphic to the rationals, and the two sided limit points of the perfect set to the irrationals. Consider the function f defined by: $f(x)=0$ if x is irrational, $f(p/q)=1/q$ for p/q in lowest terms. Then, as is easily seen, f assigns values to the rationals in an "upper semicontinuous fashion". Let $\mathscr{B}$ be the collection of fractions with even (odd) denominators. Then, there exists a perfect subset Q (resp. R) of the irrationals such that whenever x and y are distinct points of Q (resp. R), the maximum value of the function f between x and y is a rational with even (resp. odd) denominator.

The next lemma begins with a continuous nowhere constant function on a perfect set P, and constructs a perfect subset Q such that the difference quotient, $\Delta(x, y) = (f(y)-f(x))/(y-x)$, of two distinct points $x, y \in Q$ is closely approximated by the difference quotient of the endpoints of the largest interval complementary to Q between x and y.

This control over the difference quotient of points in Q plus applications of the Black and White Lemma are the basic tools which enable us to find monotonic and differentiable perfect roads in the sequel.

LEMMA 2.2. *Let f be a real continuous nowhere constant function on a perfect set P of real numbers. Suppose that the set, $\{|\Delta(x, y)| : x \neq y \in P\}$, of difference quotients on P is bounded by M. Then, there exists a nowhere dense perfect set Q in P with complementary intervals*

$$\mathscr{A} = \{(a_k^n, b_k^n) : n = 1, 2, 3, \ldots, 1 \leqq k \leqq 2^{n-1}\}$$

such that if x and y are any distinct points of Q and (a_k^n, b_k^n) is the longest interval in $\mathscr{A}$ contained in (x, y), then

$$|\Delta(x, y) - \Delta(a_k^n, b_k^n)| \leqq 7M10^{-n}.$$

Proof. Choose points $a_0, b_0 \in P^*$ with $a_0 < b_0$ and $f(a_0) \neq f(b_0)$. By continuity of f there is a $\delta < (b_0 - a_0)/10$ such that $a_0 \leqq x \leqq a_0 + \delta$ implies $|f(x) - f(a_0)| < |f(b_0) - f(a_0)|/10$, and $b_0 - \delta \leqq x \leqq b_0$, $|f(b_0) - f(x)| < |f(b_0) - f(a_0)|/10$. Choose $a_1^1 \in (a_0, a_0 + \delta) \cap P^*$ and $b_1^1 \in (b_0 - \delta, b_0) \cap P^*$ so that $f(a_0), f(b_0), f(a_1^1), f(b_1^1)$ are all distinct.

Now, for definiteness suppose $f(a_0) < f(b_0)$ and let $x \in (a_0, a_0 + \delta)$, $y \in (b_0 - \delta, b_0)$. Then, it is clear that $\Delta(x, y)$ lies between the slopes of the lines joining respectively $(a_0, f(a_0) + (f(b_0) - f(a_0))/10)$ to $(b_0, f(b_0) - (f(b_0) - f(a_0))/10)$ and $(a_0 + \delta, f(a_0) - (f(b_0) - f(a_0))/10)$ to $(b_0 - \delta, f(b_0) + (f(b_0) - f(a_0))/10)$. Computing this explicitly and using the fact that $10\delta < (b_0 - a_0)$ one obtains $\Delta(a_0, b_0)[1 - 2\cdot 10^{-1}] \leqq \Delta(x, y) \leqq \Delta(a_0, b_0)[1 + 5 \cdot 10^{-1}]$. This implies that

$$|\Delta(x, y) - \Delta(a_1^1, b_1^1)| \leqq 7\Delta(a_0, b_0)10^{-1} \leqq 7M10^{-1}.$$

By induction suppose we have determined the points $a_m^k < b_m^k$ in P^* for all $k \leqq n$ and $m \leqq 2^{k-1}$ so that the family $\mathscr{A}_n = \{(a_m^k, b_m^k) : k \leqq n, m \leqq 2^{k-1}\}$ is a collection of disjoint intervals. We proceed to define points $a_i^{n+1} < b_i^{n+1}$ for $i \leqq 2^n$. Enumerate the 2^n disjoint intervals in $[a_0, b_0] - \bigcup \mathscr{A}_n$ in natural order. Suppose (a_0', b_0') is the ith such interval. By continuity there is a $\delta < (b_0', a_0')10^{-(n+1)}$ such that $a_0' \leqq x \leqq a_0' + \delta$ implies $|f(x) - f(a_0')| \leqq |f(b_0') - f(a_0')|/10^{n+1}$ and $b_0' - \delta \leqq x \leqq b_0'$ implies $|f(b_0') - f(x)| \leqq |f(b_0') - f(a_0')|/10^{n+1}$. Select $a_i^{n+1} \in (a_0', a_0' + \delta) \cap P^*$ and $b_i^{n+1} \in (b_0' - \delta, b_0')$ such that $f(a_i^{n+1})$ and $f(b_i^{n+1})$ are distinct and distinct from the values of f at all previously defined endpoints a_m^k, b_m^k, $k \leqq n$, $m \leqq 2^{k-1}$ and $a_j^{n+1}, b_j^{n+1}, j \neq i$. This defines $\mathscr{A}_{n+1} = \{(a_m^k, b_m^k) : k \leqq n+1, m \leqq 2^{k-1}\}$. Let $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}_n$ and $Q = [a, b] - \bigcup \mathscr{A}$. Clearly, Q is a nowhere dense perfect set in P.

Let $x, y \in Q$, $x < y$. There is a longest interval (a_k^n, b_k^n) in $\mathscr{A}$ which lies in (x, y). Furthermore, it is clear from our construction that if (a_k^n, b_k^n) was constructed from the gap $[a_0', b_0']$ in the intervals $\mathscr{A}_{n-1}$, then $x \in (a_0', a_0' + (b_0' - a_0')/10^n)$ and

$$y \in (b_0', b_0' - (b_0' - a_0')/10^n).$$

By a computation similar to that performed at the first step it is easily seen that

$$|\Delta(x, y) - \Delta(a_k^n, b_k^n)| \leqq 7M10^{-n}$$

as required.

These two lemmas combine to produce a simple proof of a recent theorem of Filipczak [4]. In fact, a somewhat less involved construction in Lemma 2.2 is required. Namely, in the construction of each interval (a_k^n, b_k^n), say, from the gap $[a, b]$, one may choose $\delta < (b-a)/3$ so that $x \in [a, a+\delta]$ implies $|f(x) - f(a)| < |f(b) - f(a)|/2$ and $y \in [b-\delta, b]$ implies $|f(b) - f(y)| < |f(b) - f(a)|/2$. Then note that for such x and y, $f(y) > f(x)$ if and only if $f(b) > f(a)$.

With this remark we prove

THEOREM 2.3 (FILIPCZAK). *Let f be a continuous real function on a perfect set P of real numbers. Then, there exists a perfect set Q contained in P such that f is monotonic on Q.*

Proof. We may clearly assume f to be nowhere monotonic, otherwise there is nothing to prove. First, construct a perfect set Q' in P as in Lemma 2.2 but with the modification outlined in the remarks preceding this theorem. Re-enumerate $\mathscr{A}$ as $\{(a_m, b_m)\}_{m=1}^{\infty}$ for convenience. Since f is one-to-one on the set of endpoints of the intervals in $\mathscr{A}$, $f(b_m) - f(a_m) \neq 0$ for all m. Let $A_+ = \{a_m : f(b_m) - f(a_m) > 0\}$ and $A_- = \{a_m : f(b_m) - f(a_m) < 0\}$. Clearly, one of these sets is somewhere dense in P. Without loss of generality, assume A_+ is dense in $J \cap P$ where J is some closed interval.

Now, apply the Black and White Lemma to obtain a perfect set Q in $J \cap P$ which has the property of that lemma for $\mathscr{B} = \{(a_m, b_m) : a_m \in A_+\}$. Then, suppose $x, y \in Q$, $x < y$. Let (a_m, b_m) be the largest interval of $\mathscr{A}$ contained in (x, y). First of all, by the construction of Lemma 2.2 this largest interval was constructed from some gap $[a, b]$ in such a way that $a < x < a_m < b_m < y < b$ and with $a_m - a$ and $b - b_m$ so small that the three differences $f(b)-f(a)$, $f(b_m)-f(a_m)$, $f(y)-f(x)$ have the same sign. Secondly, by the Black and White Lemma this largest interval is in $\mathscr{B}$, i.e., $f(b_m)-f(a_m)>0$. Thus, $f(y)-f(x)>0$ and f is strictly increasing as a function on Q as required.

3. **The main theorem.** According to a well-known theorem of Lebesgue, a real valued function f defined and monotonic on an interval I of real numbers is differentiable almost everywhere on I. The requirement that I be an interval is not necessary; the domain of f can be any set of real numbers, in particular, a perfect set P. If P happens to be nowhere dense, Lebesgue's Theorem has nontrivial content if and only if P has positive measure. On the other hand, any two nowhere dense perfect sets of real numbers are homeomorphic, so one might conjecture that even if P has zero measure, there must be "many" points of differentiability. (Of course, since it is easy to define an increasing f on any perfect set of zero measure whose derivative is identically ∞, we must include the possibility of infinite derivatives in our discussion.) We begin this section (Example 3.1 below) by showing that the dependence of Lebesgue's Theorem on measure in this case is essential. However, we then employ Lemmas 2.1 and 2.2 to show (Theorem 3.2 and Corollary 3.3) that to each monotone function defined on a perfect set P there corresponds a perfect subset Q such that the restriction of f to Q is differentiable everywhere on Q. Actually, we show that this result holds under a far weaker hypothesis on f than monotonicity.

Denote Lebesgue measure on the line by λ. Recall that the upper (lower) density of a measurable set H at a point x is the upper (lower) limit of the ratios $\lambda(H\cap I)/\lambda(I)$ as $\lambda(I)\to 0$ through the family of closed intervals I containing x. The construction below uses a refinement of a construction devised by Goffman [5] for other purposes.

EXAMPLE 3.1. Let P be a zero measure perfect set of real numbers. Then, there exists an increasing real valued function on P which is nowhere differentiable.

Proof. Without loss of generality we may assume that P is bounded and contained in [0, 1]. We first construct a set H in [0, 1] whose upper and lower densities at each $x\in P$ are 1 and 0 respectively. We then define the desired function on [0, 1] and show that its restriction to P is nowhere differentiable.

Remove enough of the complementary intervals of P so that the remaining set, H_1, satisfies $\lambda(H_1)<1$. Let I be one of the closed intervals in the disjoint family of closed intervals which make up H_1. Remove enough of the complementary intervals of P from I so that the measure of the remaining portion of I is less than $\lambda(I)/2$. This portion is a disjoint union of closed intervals in I. Perform this construction for each such I and let H_2 be the union of all the closed intervals so obtained.

In general having constructed the set H_k let I be one of its closed intervals. Remove enough of the complementary intervals of P from I so that the measure of the remaining portion is less than $\lambda(I)/(k+1)$. This portion is a disjoint union of closed intervals. Perform this construction for each such interval I and let H_{k+1}

be the union of all the closed intervals so obtained. This procedure defines inductively a sequence $\{H_k\}$ of closed sets satisfying:

(1) H_k is a finite disjoint union of closed intervals.

(2) $H_1 \supset H_2 \supset \cdots$.

(3) $\bigcap_{k=1}^{\infty} H_k = P$.

(4) If I is one of the intervals making up H_k, then $\lambda(H_{k+1} \cap I) < \lambda(I)/(k+1)$.

Let $H = (H_1 - H_2) \cup (H_3 - H_4) \cup (H_5 - H_6) \cup \cdots$. Let $x \in P$ and let k be given. Then, $x \in H_k$ and for some one of the intervals, say I_k, which make up H_k we have $x \in I_k$. If k is odd, then $H \cap I_k \supset (H_k - H_{k+1}) \cap I_k$, $\lambda(H \cap I_k) \geqq \lambda((H_k - H_{k+1}) \cap I_k) = \lambda(I_k - H_{k+1}) = \lambda(I_k) - \lambda(H_{k+1} \cap I_k) > \lambda(I_k)(1-(k+1)^{-1})$ by (4). Since $\lambda(P) = 0$, as odd k tends to ∞, $\lambda(I_k) \to 0$ and $\lambda(H \cap I_k)/\lambda(I_k)$ tends to 1. Thus, H has upper density 1 at x. On the other hand suppose k is even. Then, $(H_{k-1} - H_k)$ is disjoint from I_k and by (2) the same is true of $(H_1 - H_2), \ldots, (H_{k-3} - H_{k-2})$. Thus $\lambda(H \cap I_k) = \lambda([(H_{k+1} - H_{k+2}) \cup \cdots] \cap I_k) \leqq \lambda(H_{k+1} \cap I_k) < \lambda(I)/(k+1)$ by (4). Then, as above, $\lambda(I_k) \to 0$ while $\lambda(H \cap I_k)/\lambda(I_k) \to 0$. Thus, H has lower density zero at x.

Next, define g on $[0, 1]$ by $g(x) = \lambda(H \cap [0, x])$. Clearly, g is monotone increasing. Given $0 \leqq a < b \leqq 1$, denote the difference quotient $(g(b) - g(a))/(b-a)$ by $\Delta(a, b)$. Notice that if $a \leqq x \leqq b$ and $t = (b-x)/(b-a)$, then $\Delta(a, b) = t\Delta(x, b) + (1-t)\Delta(a, x)$. That is, the collection of difference quotients at x can be at least as large or small as the difference quotients $\Delta(a, b)$ for which $a \leqq x \leqq b$. Let $x \in P$, let k be given and suppose the closed interval of H_k which contains x is $I_k = [a_k, b_k]$. We compute

$$\Delta(a_k, b_k) = \frac{\lambda(I_k \cap H)}{\lambda(I)} \begin{array}{ll} < (k+1)^{-1}, & k \text{ even} \\ > (1-(k+1)^{-1}), & k \text{ odd} \end{array}$$

by the above estimates. Thus, the upper derivative of g at x is 1 while the lower derivative of g at x is 0. But in the computation of difference quotients above only points of P were involved. Thus, the restriction of g to P is a monotone nowhere differentiable function on P as required.

It is important in the remaining theorems of this section to remember that the derivatives of a function restricted to a set Q are computed from difference quotients using points of Q only. In general, if f is defined on a set P and A is a subset of P having the point $x \in A$ as a limit point we will use the following notation:

$$\bar{D}_A f(x) = \limsup_{y \to x} \left\{ \frac{f(y) - f(x)}{y - x} : y \in A \right\},$$

$$\underline{D}_A f(x) = \liminf_{y \to x} \left\{ \frac{f(y) - f(x)}{y - x} : y \in A \right\},$$

$$\bar{D} = \bar{D}_P, \qquad \underline{D} = \underline{D}_P.$$

For simplicity we shall sometimes write $D_{1,2}$ for $D_{P_{1,2}}$, etc.

THEOREM 3.2. *Let f be a real continuous function on a perfect set P of real numbers. Then, there exists a perfect set Q contained in P such that the restriction of f to Q is differentiable on Q.*

Proof. By Filipczak's Theorem (Theorem 2.3) we can assume without loss of generality that f is strictly increasing. Then $\underline{D}f(x) \geqq 0$ and there are two possibilities for the set $A = \{x \in P : \bar{D}f(x) = \infty\}$: Case I: A is empty; Case II: A is nonempty. The proof of Case II will be similar to that of Case I.

Case I. $\{x \in P : \bar{D}f(x) = \infty\} = \varnothing$. For each n let $E_n = \{x \in P : \Delta(x, y) \leqq n$ for each $y \in P,\ y \neq x\}$. Then $P = \bigcup_{n=1}^{\infty} E_n$, otherwise some upper derivative would be infinite. Since P is perfect and each E_n is closed, some E_n must contain a relative interval of P. Therefore, we may assume without loss of generality that the set $\{\Delta(x, y) : x, y \in P, x \neq y\}$ is bounded, say, by M. If f were constant in some relative interval, we would be through; so, we may also assume that f is nowhere constant.

With these simplifications let P_0 be a perfect subset of P having the properties of Lemma 2.2. Enumerate the set $\mathscr{A}$ of complementary intervals of P_0 as $\{(a_k, b_k)\}_{k=1}^{\infty}$ and let $m_k = \Delta(a_k, b_k)$ so $|m_k| \leqq M$ for all k. Partition $[0, M]$ into finitely many disjoint half-open intervals $\{I_i\}$ of length less than $1/2$ and let $A_i = \{a_k : m_k \in I_i\}$. Clearly, at least one of these sets, say A_s, is somewhere dense in P_0. That is, there exists an interval J_1 such that $P_0 \cap J_1$ is perfect and equals the closure of $A_s \cap J_1$.

Apply the Black and White Lemma relative to the dense family of intervals $\{(a_k, b_k) : a_k \in A_s\}$. We obtain a perfect set P_1 in $P_0 \cap J_1$ such that $x, y \in P_1$ and (a_k, b_k) is the largest interval of $\mathscr{A}$ between x and y, then $m_k \in A_s$.

We claim that $|\bar{D}_1 f(x) - \underline{D}_1 f(x)| \leqq 1/2$ for all $x \in P_1$. Let $\varepsilon > 0$ and let $x \in P_1$. Choose sequences $\{y_n\}$ and $\{z_n\}$ approaching x for which $|\bar{D}_1 f(x) - \Delta(x, y_n)| < \varepsilon/n$, and $|\underline{D}_1 f(x) - \Delta(x, z_n)| < \varepsilon/n$. Let $(a(y_n), b(y_n))$ (respectively, $(a(z_n), b(z_n))$) denote the largest interval of $\mathscr{A}$ between x and y_n (respectively, x and z_n). Referring to the proof of Lemma 2.2, suppose $(a(y_n), b(y_n)) \in \mathscr{A}_{k(n)}$ and $(a(z_n), b(z_n)) \in \mathscr{A}_{j(n)}$. Clearly, $k(n)$ and $j(n)$ tend to infinity with n. Using the estimates of Lemma 2.2 we estimate

$$\begin{aligned} |\bar{D}_1 f(x) - \underline{D}_1 f(x)| &\leqq |\bar{D}_1 f(x) - \Delta(x, y_n)| + |\underline{D}_1 f(x) - \Delta(x, z_n)| \\ &\quad + |\Delta(x, y_n) - m_{k(n)}| + |\Delta(x, z_n) - m_{j(n)}| + |m_{k(n)} - m_{j(n)}| \\ &\leqq 2\varepsilon/n + 7M/10^{k(n)} + 7M/10^{j(n)} + 1/2 \end{aligned}$$

and letting n tend to ∞ our claim is supported.

Let α and β be the extreme endpoints of P_1 and choose $\gamma \in (\alpha, \beta) \cap P_1$ such that (α, γ) and (γ, β) both intersect P_1. Let $\mathscr{M} = \{m_k : (a_k, b_k) \subset (\alpha, \gamma)\}$. Proceeding as in the first step, we partition $[0, M]$ into disjoint half-open intervals of length less than $1/4$. Then, there exists a subfamily $\mathscr{M}'$ of diameter less than $1/4$ such that $\{a_k : m_k \in \mathscr{M}'\}$ is dense in some subinterval $J_{1,1}$ of $(\alpha, \gamma) \cap P_1$. Just as before we apply the Black and White Lemma to obtain a perfect set $P_{1,1}$ in P_1 such that if $x, y \in P_{1,1}$ and (a_k, b_k) is the largest interval of $\mathscr{A}$ between x and y, then $m_k \in \mathscr{M}'$. Repeating the above estimates we obtain $|\bar{D}_{1,1} f(x) - \underline{D}_{1,1} f(x)| \leqq 1/4$ for all $x \in P_{1.1}$.

Likewise, in (γ, β) we can find a perfect set $P_{1,2} \subset P_1$ such that

$$|\bar{D}_{1,2} f(x) - \underline{D}_{1,2} f(x)| \leqq 1/4 \quad \text{for } x \in P_{1,2}.$$

In general, we continue this process of splitting the perfect sets $P_{1,1}$, $P_{1,2}$, etc., in half and partitioning $[0, M]$ into pieces of length less than $1/2^k$ (at the kth stage). We obtain a family of nonvoid perfect sets $\{P_{n_1,n_2,\ldots,n_k} : n \in N, k=1, 2, \ldots\}$ where N denotes the collection of all sequences of 1's and 2's for which $n_1 = 1$. The family satisfies

$$P_{n_1,n_2,\ldots,n_k,i} \subset P_{n_1,n_2,\ldots,n_k}, \qquad i = 1, 2,$$
$$|\bar{D}_{n_1,\ldots,n_k} f(x) - \underline{D}_{n_1,\ldots,n_k} f(x)| \leqq 1/2^k.$$

Define $Q = \bigcap_{k=1}^{\infty} \bigcup \{P_{n_1,n_2,\ldots,n_k} : n \in N\}$. As in the proof of Lemma 2.1, Q is a (nonempty) perfect subset of P. Furthermore, for each $x \in Q$ and each k

$$|\bar{D}_Q f(x) - \underline{D}_Q f(x)| \leqq |\bar{D}_{n_1,\ldots,n_k} f(x) - \underline{D}_{n_1,\ldots,n_k} f(x)| \leqq 1/2^k.$$

Thus, the restriction of f to the perfect set Q is differentiable as required.

Case II. $\{x \in P : \bar{D}f(x) = \infty\}$ is nonempty. Let P_1 be the perfect set obtained from P by removing the set of intervals $\mathscr{A} = \{(a_k, b_k)\}_{k=1}^{\infty}$ as constructed in Lemma 2.2. Define $A_1 = \{a_k : m_k \leqq 1\}$, $C_1 = \{a_k : m_k > 1\}$. One of these sets is somewhere dense in P_1.

If A_1 is dense in P_1 in some interval J, we can apply the Black and White Lemma to obtain a perfect set $Q \subset P_1 \cap J$ such that if $x, y \in Q$ and (a_k, b_k) is the largest interval of $\mathscr{A}$ between x and y, then $m_k \leqq 1$. From the inequality in the proof of Lemma 2.2 we have $|\Delta(x, y) - m_k| \leqq 7 \cdot 10^{-n} \Delta(c, d) < \Delta(c, d)$ where (c, d) is the gap from which (a_k, b_k) was constructed in that proof. We also have $|\Delta(c, d) - m_k| \leqq 7 \cdot 10^{-n} \Delta(c, d)$ so $\Delta(c, d) < m_k 10^n/(10^n - 7) \leqq 4$.

Thus, $|\Delta(x, y)| < 4 + m_k \leqq 5$. Therefore, we can apply Case I to f and Q to obtain the result.

On the other hand, if C_1 is dense in P_1 in some interval J, we can imitate the proof of Case I by splitting $J \cap P_1$ in two and find perfect sets $P_{1,1}$, $P_{1,2}$ in P_1 having the property that $x, y \in P_{1,i}$ implies $m_k > 1$ where (a_k, b_k) is the longest interval of $\mathscr{A}$ between x and y.

Now, consider $P_{1,1}$ and the sets $A_{1,1} = \{a_k : 1 < m_k \leqq 2\}$ and $C_{1,1} = \{a_k : m_k > 2\}$. If $A_{1,1}$ is somewhere dense in $P_{1,1}$, we are as above reduced to Case I. If $C_{1,1}$ is somewhere dense we can again obtain two perfect sets $P_{1,1,1}$, $P_{1,1,2}$ in $P_{1,1}$ such that $x, y \in P_{1,1,i}$ implies $m_k > 2$ where (a_k, b_k) is the largest interval of $\mathscr{A}$ between x and y.

We continue this inductive process in the obvious way. If at any stage some $A_{n_1,\ldots,n_k}$ is somewhere dense in $P_{n_1,\ldots,n_k}$, we are reduced to Case I and the proof is completed. Let us assume this is never the case. Then, we have obtained a family of perfects sets $\{P_{n_1,\ldots,n_k} : n \in N, k=1, 2, \ldots\}$ where N is as in Case I. We have

(1) $P_{n_1,\ldots,n_k,i} \subset P_{n_1,\ldots,n_k}$ for $i=1, 2$ and each k.

(2) If $x, y \in P_{n_1,\ldots,n_k}$, and (a_k, b_k) is the largest interval of $\mathscr{A}$ between x and y, then $m_k > k$.

Let $Q = \bigcap_{k=1}^{\infty} \bigcup \{P_{n_1,\ldots,n_k} : n \in N\}$. As in Lemma 2.1, Q is nonempty and perfect. Let $x \in Q$ be fixed and let $y \in Q$. Then, there exists $n \in N$ and an integer $i(y)$

such that $x, y \in P_{n_1,\ldots,n_{i(y)}}$ while $y \notin P_{n_1,\ldots,n_{i(y)},n_{i(y)}+1}$. Let (a_k, b_k) be the largest interval of $\mathscr{A}$ between x and y. Then, $m_k > i(y)$ and

$$|\Delta(x, y)| \geqq \Delta(a_k, b_k)\frac{b_k-a_k}{y-x} \geqq i(y)\frac{b_k-a_k}{d-c} \geqq i(y)\frac{8}{10},$$

where (a_k, b_k) was constructed in the proof of Lemma 2.2 from the gap (c, d). But $y \to x$ implies $i(y) \to \infty$; so $\lim_{y \to x} \Delta(x, y) = \infty$. Consequently, $D_Q f(x) = \infty$ for all $x \in Q$ and f is differentiable on Q in the extended sense.

Recall that a function f has the property of Baire on a set if for each open subset G of the range there is an open set G' and two sets M_1, M_2 of first category such that $f^{-1}(G) = (G' - M_1) \cup M_2$. In particular any Borel function has the property of Baire [9, p. 306].

COROLLARY 3.3. *Let f have the property of Baire on an uncountable set A of type G_δ. Then, there is a perfect set Q in A such that the restriction of f to Q is differentiable.*

Proof. Since f has the property of Baire there exists a residual subset B of A such that the restriction of f to B is continuous. The set B in turn contains a perfect set and Theorem 3.2 applies to give the result.

We see from the proof of Theorem 3.2 that the derivative of the restriction of f to Q is either bounded or identically infinite. If we include the identically infinite function among the differentiable functions with derivative the identically zero function, we may prove that Q may be chosen in such a way that f is infinitely differentiable when restricted to Q. For this we need the result of Hájek [6] which states that the upper derivative of an arbitrary function defined on an interval is Borel measurable of class 2, and therefore has the property of Baire. We claim that this same theorem holds for a continuous function, f, defined on a perfect set P. For, given such an f we may extend it by linearity on the complementary intervals of P to be continuous everywhere. By Hájek's theorem, $\bar{D}f$ is in Borel class 2 and thus is in Borel class 2 when restricted to P. But $\bar{D}_P f$ differs from $\bar{D}f$ only at the one-sided limit points of P; namely, only on a countable set. Consequently, $\bar{D}_P f$ is Borel class 2 as claimed. We use this remark in the proof of the following theorem.

THEOREM 3.4. *Let f have the property of Baire on an uncountable set A of type G_δ. Then, there is a perfect set Q in A such that the restriction of f to Q is infinitely differentiable.*

Proof. By Corollary 3.3, f is differentiable on a perfect set P in A. If $f' \equiv \infty$ on P, then f is infinitely differentiable on P. Otherwise, by the remarks preceding this theorem f' is bounded on P. Let Q_1, Q_2 be nonempty disjoint perfect subsets of P with diameter $\leqq 1$. Then by the above extension of Hájek's theorem f' is a Borel function on each of the perfect sets Q_1, Q_2 and so by Corollary 3.3 there are perfect sets $P_i \subset Q_i$, $i = 1, 2$, on which f' is differentiable, i.e., on which f is twice differentiable. Should the occasion arise that on P_1 or P_2, $f'' \equiv \infty$ we are through. Otherwise, repeat the above process to each of P_1 and P_2 to obtain disjoint perfect sets

$P_{1,1}, P_{1,2} \subset P_1$, $P_{2,1}, P_{2,2} \subset P_2$ of diameter $\leqq 1/2$ on each of which f''' exists. Continuing, we obtain a system of nonempty perfect sets $\{P_{n_1,n_2,\ldots,n_k} : n \in N\}_{k=1}^{\infty}$ where N is all sequences of 1's and 2's, diameter of $P_{n_1,n_2,\ldots,n_k} \leqq 1/k$, and $f^{(k+1)}$ exists on $\bigcup \{P_{n_1,n_2,\ldots,n_k} : n \in N\}$. If for some $n \in N, f^{(k+1)} \equiv \infty$ on $P_{n_1,\ldots,n_k}$ we are through. Failing this the set $Q = \bigcap_{k=1}^{\infty} \bigcup \{P_{n_1,\ldots,n_k} : n \in N\}$ is a nonempty perfect set in A such that f is (boundedly) infinitely differentiable on Q.

4. **On extensions of Theorem 3.2.** Theorem 3.2 suggests certain questions which we consider in this section.

We observe first that the hypothesis of Theorem 3.2 can be weakened considerably. For example, if P is a perfect set and f possesses the property of Baire on P, then there is a set R residual in P such that $f|R$ is continuous. The set R contains a perfect subset S. Similarly, let f be any function defined on a perfect set P and measurable with respect to a measure μ which vanishes on singleton sets such that $\mu(P) > 0$. If Lusin's theorem holds for μ, then there exists a closed set K such that $\mu(K) > 0$ and $f|K$ is continuous. Since denumerable sets have μ-measure zero, K contains a perfect set S.

Thus, in either case, there exists a perfect set S such that $f|S$ is continuous. We now apply Theorem 3.2 to obtain a perfect set Q such that $f|Q$ is differentiable. We sum these remarks up as a theorem.

THEOREM 4.1. *Let f be defined on a perfect set P. Suppose f satisfies either condition* (i) *or condition* (ii) *below*:

(i) *f has the property of Baire on P*;

(ii) *f is measurable with respect to a nonatomic measure μ for which Lusin's theorem is valid and such that $\mu(P) > 0$.*

Then there exists a perfect subset Q of P such that $f|Q$ is differentiable.

We remark that condition (ii) of Theorem 4.1 is satisfied in particular by any continuous Lebesgue-Stieltjes measure μ such that $\mu(P) > 0$.

In 1936, Maximoff [10] obtained several theorems involving the continuity structure of functions possessing certain properties. Some of these theorems can be extended to give information concerning the differentiability structure of functions possessing these properties. For example, one of Maximoff's results states that if f is a Borel measurable function defined on an interval I then there exists a denumerable set $B \subset I$ such that to each point $x_0 \in I \sim B$ there corresponds a perfect set Q containing x_0 as a two-sided limit point such that $f|Q$ is continuous.

THEOREM 4.2. *Let f be Borel measurable on* $[a, b]$. *There exists a denumerable set D such that to each x in the complement of D there corresponds a perfect set P having x as a two-sided limit point such that $f|P$ is differentiable.*

Proof. Assume first that f is bounded. According to Maximoff's Theorem, there exists a denumerable set B such that if $x \in [a, b] \sim B$, then there exists a perfect set P such that P has x as a two-sided limit point, and $f|P$ is continuous. Define a function

$\hat{f}$ as follows: For $x \in [a, b] \sim B$, let $\hat{f}(x) = f(x)$, and for $x \in B$, let $\hat{f}(x) = \lim\sup f(t)$, this limit superior being computed for $t \to x$, $t \in [a, b] \sim B$. Now an immediate consequence of a theorem of Bagemihl [1] asserts that for any function, the sets of left and right derived numbers at a point x have a point in common, except for x in a certain denumerable set. Let A be this exceptional set for $\hat{f}$. Let $D = A \cup B$ and let $x_0 \in [a, b] \sim D$. Let α be an extended real number which is both a right derived number and a left derived number for $\hat{f}$ at x. Let $\{y_n\}$ and $\{z_n\}$ be two sequences of numbers, the first increasing to x_0, the second decreasing to x_0 such that

$$\lim_{n\to\infty} \frac{\hat{f}(y_n) - \hat{f}(x_0)}{y_n - x_0} = \lim_{n\to\infty} \frac{\hat{f}(z_n) - \hat{f}(x_0)}{z_n - x_0} = \alpha.$$

Among the points in these sequences, some might be in B. If, say, y_k is such a point, and $\varepsilon > 0$, there exists a point $y_k' \in [a, b] \sim B$ such that $|y_k - y_k'| < \varepsilon$ and

$$\left| \frac{\hat{f}(y_k') - \hat{f}(x_0)}{y_k' - x_0} - \frac{\hat{f}(y_k) - \hat{f}(x_0)}{y_k - x_0} \right| < \varepsilon$$

provided α is finite. The analogous statement holds if α is infinite. The existence of such a point y_k' follows directly from the definition of $\hat{f}$. Since such replacements are possible when necessary, we may assume without loss of generality that the members of the sequences $\{y_k\}$ and $\{z_k\}$ are all in the set $[a, b] \sim B$. On this set, $f = \hat{f}$. We thus have obtained two sequences $\{y_k\}$ and $\{z_k\}$ such that $y_k \uparrow x_0$, $z_n \downarrow x_0$, and

$$\lim_{k\to\infty} \frac{f(y_k) - f(x_0)}{y_k - x_0} = \alpha = \lim_{k\to\infty} \frac{f(z_k) - f(x_0)}{z_k - x_0}$$

and none of the points in these sequences is in B.

It follows from Maximoff's Theorem that for each $k = 1, 2, \ldots$ there exists a perfect set P_k containing y_k as a two-sided limit point such that $f|P_k$ is continuous. The sets P_k can be so chosen that they are contained in pairwise disjoint open intervals and so that if $t \in P_k$ then

$$\left| \frac{f(t) - f(x_0)}{t - x_0} - \frac{f(y_k) - f(x_0)}{y_k - x_0} \right| < \frac{1}{k}.$$

We now apply Theorem 3.2 to each set P_k obtaining a perfect set $Q_k \subset P_k$ such that $f|Q_k$ is differentiable. If $Q_1 = \{x_0\} \cup \bigcup_{k=1}^{\infty} Q_k$ then $f|Q_1$ is differentiable with (right) derivative α at x_0. In a similar way we obtain a perfect set Q_2 having x_0 as a left limit point such that $f|Q_2$ is differentiable with (left) derivative α at x_0.

The set $Q = Q_1 \cup Q_2$ has all the properties required by the conclusion of Theorem 4.2. The proof is therefore complete for f bounded. To see that the theorem is still valid if f is unbounded we need only consider the function $g(x) = \arctan f(x)$. This function is bounded, and therefore has a (two-sided) perfect differentiable road at all x not in some denumerable set. It is easy to verify that if P is such a (two-sided) perfect differentiable road at x, then P is also a (two-sided) perfect differentiable road for $f(x) = \tan g(x)$. The proof of Theorem 4.2 is now complete.

We note now that it is not enough to assume merely that f be (Lebesgue) measurable or that f have the property of Baire in the statement of Theorem 4.2. To see this, let K be the Cantor set and let S be any totally imperfect subset of K; that is, any subset S having the cardinality of the continuum but containing no perfect set. Let f be the characteristic function of S. It is clear that f is Lebesgue measurable and has the property of Baire but has no two-sided perfect differentiable road at any point of S.

Maximoff showed in [10] that if one assumes that f also has the Darboux property, then the exceptional set B appearing in the statement of his theorem is empty. No analogous statement is possible for two-sided perfect differentiable roads. In fact, given any denumerable set B, there is a continuous function f whose left and right derivatives exist but are different at each point of B. It is clear that no (two-sided) perfect differentiable road can exist at such points.

Theorems 2.3 and 3.2 suggest the existence of other types of roads for functions continuous on a perfect set P. For example, we might ask whether every such function has a convex or concave road. That is, must there exist a perfect set $Q \subset P$ such that $f|Q$ is convex or concave. Example 4.3 below shows that such roads do not always exist.

EXAMPLE 4.3. Let f be strictly increasing on [0, 1] with $f'(x)=0$ a.e. Let $A = \{x : f'(x)=0\}$ and let P be any perfect subset of A. The set P is nowhere dense, f is strictly increasing on P, and the derivative of $f|P$ vanishes identically. It is easy to verify that P contains no perfect subset on which f is convex or concave. In fact, P contains no dense-in-itself subset (even denumerable!) on which f is convex or concave.

Theorem 3.2 asserts that every function continuous on a perfect set P contains a perfect differentiable road. One might suspect that if P is an interval one can infer the existence of "larger" differentiable roads. Examples 4.4 and 4.5 show that to a certain extent Theorem 3.2 is the best possible. These examples show that there exist continuous functions for which every differentiable road is nowhere dense, and there exist functions for which no differentiable road has positive measure. Example 4.4 may be contrasted with a theorem of Blumberg's [3] according to which *every* function f possesses a dense set D such that $f|D$ is continuous.

EXAMPLE 4.4. Besicovitch [2] (see also [8]) has given an example of a function f, continuous on [0, 1] and having at no point a derivative even one-sided, finite or infinite. If there were a dense set D such that $f|D$ is differentiable (even one-sided), then the continuity of f would imply that f (considered as a function on [0, 1]) is differentiable on D.

EXAMPLE 4.5. Jarnik [7] (see also [12]) has given an example of a function f continuous on [0, 1] with the property that for almost all x, the upper bilateral approximate derivate at x is ∞ while the lower bilateral approximate derivate at x is $-\infty$. In particular, f is almost nowhere approximately derivable in the extended sense. Let M be any set of positive measure. Let $A=\{x : f'_{ap}$ does not exist in the extended sense at $x\}$. Since A has full measure, the set $M \cap A$ has positive measure. Let B consist of the points in $M \cap A$ which are points of density of $M \cap A$. By Lebesgue's Density Theorem, B has positive measure. Now, if $f|M$ is differentiable, the same would be true of $f|B$. But $f|B$ must be nowhere differentiable, for if $f|B$ were differentiable at x_0, then f would be approximately differentiable at x_0.

We mention in passing that both the Besicovitch type of function and the Jarnik type of function are exceptional in that the set of all such functions is of the first category in $C[0, 1]$ (see [11], [7]). In fact, Jarnik proved that "most" continuous functions have a property which in one sense indicates even more pathology than his example, but in anothei sense indicates more regularity.

JARNIK'S THEOREM. *Let S be the subset of $C[a, b]$ defined as follows: $f \in S$ if and only if there exists a set $A \subset [0, 1]$ of measure zero such that for $x \in [0, 1] \sim A$, every real number is an essential right derived number of f at x. Then S is residual in $C[a, b]$.*

(The number λ is said to be an essential right derived number of f at x provided there exists a set B having unit right upper density at x such that $\lim [f(t)-f(x)]/(t-x)=\lambda$ for $t \to x$, $t \in B$.)

It is clear that no differentiable road for such a function can have positive measure. Thus the set of functions in $C[a, b]$ with no differentiable roads of positive measure is residual in $C[a, b]$.

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